MAP PROJECTIONS, MATHEMATICS AND COMPUTERS

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Abstract

Mathematics and computers are the indispensable allies of map projection. Indeed it would be beyond the abilities of today's geospatial professionals to produce a map (projection) without a computer. And embedded in the software will be mathematical algorithms. Without maths and computers, teaching map projections is reduced to imagination and hand-waving; but with them, students (and staff) can extend theory into deep practical knowledge giving them the tools to develop or improve computer software.

To demonstrate the connections we describe a set of equations developed by L. Krueger in 1912 for the transverse Mercator (TM) projection of the ellipsoid. These equations are undergoing a renaissance due entirely to Computer Algebra Systems (CAS) that have extended them so that computed errors in position are less than 5 nanometres within 3900 km of the central meridian of the projection. These extended series should completely replace current TM formula that are unnecessarily complicated and if used beyond their limited range are wildly inaccurate. We show how these series may be derived using Maxima (an early CAS) and hopefully demonstrate that the tedium of mathematical development can be side-stepped.

Keywords: Map projection, transverse Mercator, Gauss-Krueger, Karney-Krueger equations.

Biography of Author Rod Deakin started work in 1968 (age 17) as a surveyor's assistant with John Horne of Frankston, Victoria and fell in the Kananook Creek on his first day. In 1976, he graduated from the RMIT and returned to surveyor Horne's employ until 1980. In 1981, he was appointed as a tutor in surveying and then a lecturer (1983) at RMIT where he has remained. He has lectured in all aspects of surveying and in 2004 was awarded the Francis Ormond medal (RMIT University medal in honour of the founder), and in the Vice Chancellor's address at the presentation it was noted that:

> *"Indeed, Rod even holds the honour of being cited in student evaluations at another University as the "best lecturer I have had throughout my course at Melbourne University", for a series of guest lectures he provided."*

Rod regards this as his finest achievement.

Introduction

The ellipsoidal transverse Mercator (TM) projection – a conformal mapping from the ellipsoid to the plane – is widely used in the geospatial community and is also known as the Gauss-Krueger projection acknowledging C.F. Gauss's original development of the ellipsoidal form of the projection and the work of L. Krueger (1912) who re-evaluated both Gauss' work and also the contributions by Oscar Schreiber who used a simplified form of Gauss's projection for the Prussian Land Survey of 1876-1923. Krueger published two sets of equations for the transformations between the ellipsoid and the TM projection; one set (also known as Redfearn's or Thomas's equations, (Redfearn 1948, Thomas 1952)) only accurate within a narrow band of longitude about a central meridian and another more versatile set that offer micrometre accuracy anywhere within 30º of the central meridian. These latter equations, that are far more useful to the geospatial community, have been re-evaluated and improved by Poder & Engsager (1998), Engsager & Poder (2007) and Karney (2011) and are hereinafter described as the *Karney-Krueger equations* to avoid confusion with other sets of TM projection equations.

Deakin *et al.* (2010, 2011) also provide a development of the Karney-Krueger equations and show how, in the forward transformation $\phi,\lambda \to E,N$, they represent a triple projection in two parts: the first part is a conformal mapping from the ellipsoid to a sphere (conformal sphere) followed by a conformal mapping from the sphere to the plane using the spherical TM projection equations with spherical latitude ϕ replaced by conformal latitude ϕ'. This two-step process is also known as the Gauss-Schreiber projection and the scale along the central meridian is not constant. The second part is the conformal mapping from the Gauss-Schreiber plane to the TM plane where the scale factor along the central meridian is made constant.

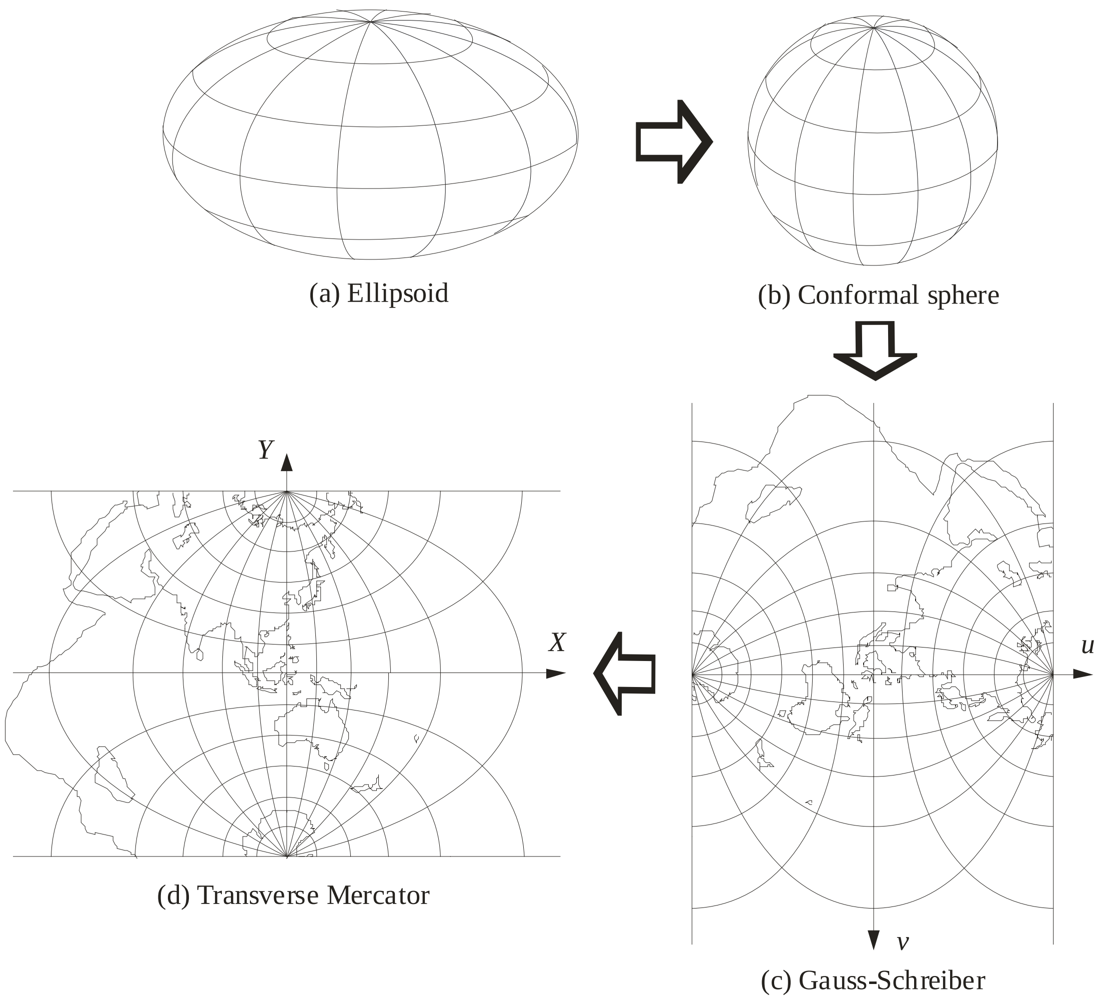

Figure 1 Karney-Krueger equations: sequence of conformal mappings
Ellipsoid → conformal sphere → Gauss-Schreiber plane → transverse Mercator plane.

The development of the Karney-Krueger equations requires some understanding of conformal mapping, complex functions, Taylor's theorem, reversion of a series, hyperbolic functions, and of course differentiation and integration. And not forgetting a basic understanding of the geometry of the reference ellipsoid, the sphere and spherical trigonometry. This is indeed 'a lot of mathematics', but in practice it may simply be using 'standard results' found in textbooks on mathematics, geodesy and map projections. In this paper, where space permits, some explanation beyond simple statements will be provided and useful information on this topic is given in Deakin et al. (2010). But the concentration will be on the use of the computer algebra system Maxima (2011) to derive a series whose coefficients α_{2k} that are key to the forward transformation $\phi,\lambda \to E,N$ of the Karney-Krueger equations. We call this series 'the α-series'.

[Note that the Karney-Krueger equations allow both forward and reverse transformations ($\phi,\lambda \Leftrightarrow E,N$) between the ellipsoid and the plane as well as computation of point scale factor and grid convergence. In this paper we will only discuss the forward transformation $\phi,\lambda \to E,N$.]

The Karney–Krueger equations

Nomenclature

α_{2k}	coefficients
ε	eccentricity of ellipsoid
ε^2	eccentricity of ellipsoid squared
η	transverse Mercator ratio $\eta = X/A$
η'	Gauss-Schreiber ratio $\eta' = v/a$
λ	longitude
λ_0	longitude of central meridian
ξ	transverse Mercator ratio $\xi = Y/A$
ξ'	Gauss-Schreiber ratio $\xi' = u/a$
σ	function of latitude
ϕ	latitude
ϕ'	conformal latitude
ω	longitude difference: $\omega = \lambda - \lambda_0$
A	rectifying radius
a	semi-major axis of ellipsoid and radius of conformal sphere
E	east grid coordinate
E_0	false origin offset
f	flattening of ellipsoid
m_0	central meridian scale factor
N	north grid coordinate
N_0	false origin offset
n	3rd flattening of ellipsoid
u	Gauss-Schreiber coordinate (north)
v	Gauss-Schreiber coordinate (east)
X	transverse Mercator coordinate (east)
Y	transverse Mercator coordinate (north)

Forward transformation: $\phi, \lambda \to E, N$ given $a, f, \lambda_0, m_0, E_0, N_0$

1. Compute ellipsoid constants $\varepsilon^2 = f\left(2-f\right)$ and $n = f/\left(2-f\right)$
2. Compute the rectifying radius A from

$$A = \frac{a}{1+n}\left\{1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6 + \cdots\right\} \tag{1}$$

3. Compute conformal latitude ϕ' from

$$\tan\phi' = \tan\phi\sqrt{1+\sigma^2} - \sigma\sqrt{1+\tan^2\phi} \quad \text{where} \quad \sigma = \sinh\left\{\varepsilon\tanh^{-1}\left(\varepsilon\tan\phi\Big/\sqrt{1+\tan^2\phi}\right)\right\} \tag{2}$$

4. Compute longitude difference $\omega = \lambda - \lambda_0$

5. Compute the u,v Gauss-Schreiber coordinates from

$$u = a\tan^{-1}\left(\tan\phi'/\cos\omega\right) \qquad v = a\sinh^{-1}\left(\sin\omega\Big/\sqrt{\tan^2\phi' + \cos^2\omega}\right) \tag{3}$$

6. Compute the Gauss-Schreiber ratios $\eta' = v/a$ and $\xi' = u/a$
7. Compute the coefficients $\left\{\alpha_{2k}\right\}$ for $k = 1, 2, \ldots, 6$ from

$$\begin{aligned}
\alpha_2 &= \frac{1}{2}n - \frac{2}{3}n^2 + \frac{5}{16}n^3 + \frac{41}{180}n^4 - \frac{127}{288}n^5 + \frac{7891}{37800}n^6 + \cdots & \alpha_4 &= \frac{13}{48}n^2 - \frac{3}{5}n^3 + \frac{557}{1440}n^4 + \frac{281}{630}n^5 - \frac{1983433}{1935360}n^6 + \cdots \\
\alpha_6 &= \frac{61}{240}n^3 - \frac{103}{140}n^4 + \frac{15061}{26880}n^5 + \frac{167603}{181440}n^6 - \cdots & \alpha_8 &= \frac{49561}{161280}n^4 - \frac{179}{168}n^5 + \frac{6601661}{7257600}n^6 + \cdots \\
\alpha_{10} &= \frac{34729}{80640}n^5 - \frac{3418889}{1995840}n^6 + \cdots & \alpha_{12} &= \frac{212378941}{319334400}n^6 -
\end{aligned} \tag{4}$$

8. Compute X,Y transverse Mercator coordinates (to order n^6) from

$$X = A\left\{\eta' + \sum_{k=1}^{6}\alpha_{2k}\cos 2k\xi'\sinh 2k\eta'\right\} \qquad Y = A\left\{\xi' + \sum_{k=1}^{6}\alpha_{2k}\sin 2k\xi'\cosh 2k\eta'\right\} \tag{5}$$

9. Compute E,N grid coordinates $E = m_0X + E_0$ and $N = m_0Y + N_0$

Derivation of the α-series coefficients

The α-series coefficients in the Karney-Krueger equations are actually the coefficients in a series that has the rectifying latitude μ as a function of the conformal latitude ϕ'. How this series is derived using Maxima and how it is a part of the conformal mapping sequence that is the ellipsoidal TM projection is set out below. And as a necessary part of the sequence, definitions and equations are given for the ellipsoid and associated constants; meridian distance *M* and quadrant length *Q* of the ellipsoid; rectifying latitude μ; rectifying radius *A* and conformal latitude ϕ'.

Ellipsoid

The ellipsoid is a surface of revolution created by rotating an ellipse (whose semi-axes are *a* and *b* and $a > b$) about its minor axis. It is the mathematical approximation of the earth and has geometric constants: flattening *f;* third flattening *n;* eccentricity ε*;* second eccentricity ε' and polar radius of curvature *c* given by

$$f = (a-b)/a \tag{6}$$

$$n = (a-b)/(a+b) = f/(2-f) \tag{7}$$

$$\varepsilon^2 = \left(a^2-b^2\right)/a^2 = f(2-f) = 4n/(1+n)^2 \tag{8}$$

$$\varepsilon'^2 = \left(a^2-b^2\right)/b^2 = \varepsilon^2/\left(1-\varepsilon^2\right) = 4n/(1-n)^2 \tag{9}$$

$$c = a^2/b = a(1+n)/(1-n) \tag{10}$$

The ellipsoid radii of curvature ρ (meridian plane) and υ (prime vertical plane) at a point whose latitude is ϕ are (Deakin & Hunter 2010a)

$$\rho = \frac{a\left(1-\varepsilon^2\right)}{\left(1-\varepsilon^2\sin^2\phi\right)^{3/2}} = \frac{a\left(1-\varepsilon^2\right)}{W^3} = \frac{c}{V^3} \qquad \text{and} \qquad \upsilon = \frac{a}{\left(1-\varepsilon^2\sin^2\phi\right)^{1/2}} = \frac{a}{W} = \frac{c}{V} \tag{11}$$

where the latitude functions *V* and *W* are

$$W^2 = 1-\varepsilon^2\sin^2\phi \qquad \text{and} \qquad V^2 = 1+\varepsilon'^2\cos^2\phi = \frac{1+n^2+2n\cos 2\phi}{(1-n)^2} \tag{12}$$

Meridian distance M

Meridian distance *M* is defined as the arc of the meridian ellipse from the equator to the point of latitude ϕ

$$M = \int_0^\phi \rho\, d\phi = \int_0^\phi \frac{a\left(1-\varepsilon^2\right)}{W^3}\, d\phi = \int_0^\phi \frac{c}{V^3}\, d\phi \tag{13}$$

This is an elliptic integral that cannot be expressed in terms of elementary functions; instead, the integrand is expanded by into a series using Taylor's theorem (see Appendix A) then evaluated by term-by-term integration. The usual form of the series formula for *M* is a function of ϕ and powers of ε^2; but the German geodesist F.R. Helmert (1880) gave a formula for meridian distance as a function of ϕ and powers of *n* that required fewer terms for the same accuracy. Helmert's method of development is given in Deakin & Hunter (2010a) and with some algebra we may write

$$M = \frac{a}{1+n}\int_0^\phi \left(1-n^2\right)^2\left(1+n^2+2n\cos 2\phi\right)^{-3/2} d\phi \tag{14}$$

We will show, using Maxima, that (14) can easily be evaluated and *M* written as

$$M = \frac{a}{1+n}\left\{c_0\phi + c_2\sin 2\phi + c_4\sin 4\phi + c_6\sin 6\phi + c_8\sin 8\phi + \cdots\right\} \tag{15}$$

where the coefficients $\{c_n\}$ are to order n^4 as follows

$$c_0 = 1+\frac{1}{4}n^2+\frac{1}{64}n^4+\cdots \qquad c_2 = -\frac{3}{2}n+\frac{3}{16}n^3+\cdots \qquad c_4 = \frac{15}{16}n^2-\frac{15}{64}n^4-\cdots$$
$$c_6 = -\frac{35}{48}n^3+\cdots \qquad c_8 = \frac{315}{512}n^4-\cdots \tag{16}$$

Quadrant length Q

The quadrant length of the ellipsoid Q is the length of the meridian arc from the equator to the pole and is obtained from equation (15) by setting $\phi = \frac{1}{2}\pi$, and noting that $\sin 2\phi, \sin 4\phi, \sin 6\phi, \ldots$ all equal zero, giving

$$Q = \frac{a\pi}{2(1+n)}\{c_0\} = \frac{a\pi}{2(1+n)}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\cdots\right\} \tag{17}$$

Rectifying latitude μ and rectifying radius A

If the meridian distance M on the ellipsoid is equivalent to a meridian distance (great circle arc) on a (rectifying) sphere of radius A then the rectifying latitude μ is defined by

$$M = A\mu \tag{18}$$

An expression for A is obtained by considering the case when $\mu = \frac{1}{2}\pi$ and M is equal to the quadrant distance Q and (18) may be rearranged to give $A = 2Q/\pi$ and then using (17) to give A to order n^4 as

$$A = \frac{a}{1+n}\{c_0\} = \frac{a}{1+n}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\cdots\right\} \tag{19}$$

Re-arranging (18) and using (15) and (19) gives the rectifying latitude as

$$\begin{aligned}\mu &= M/A \\ &= \phi+(c_2/c_0)\sin 2\phi+(c_4/c_0)\sin 4\phi+(c_6/c_0)\sin 6\phi+(c_8/c_0)\sin 8\phi+\cdots \\ &= \phi+d_2\sin 2\phi+d_4\sin 4\phi+d_6\sin 6\phi+d_8\sin 8\phi+\cdots\end{aligned} \tag{20}$$

where the coefficients $\{d_n\}$ are to order n^4

$$d_2 = -\frac{3}{2}n+\frac{9}{16}n^3-\cdots \qquad d_4 = \frac{15}{16}n^2-\frac{15}{32}n^4+\cdots \qquad d_6 = -\frac{35}{48}n^3+\cdots \qquad d_8 = \frac{315}{512}n^4-\cdots \tag{21}$$

Maxima can easily perform the algebra of (20) by using a Taylor series representation of c_0^{-1}.

Conformal latitude ϕ'

The conformal mapping of the ellipsoid onto a sphere of radius a having orthogonal curvilinear coordinates ϕ', λ (parallels and meridians) yields the differential relationship (Deakin et al. 2010)

$$\frac{d\phi'}{\cos\phi'} = \frac{\rho\, d\phi}{\upsilon\cos\phi} \tag{22}$$

Using (11) and (12) in the right-hand-side of (22) we may write

$$\frac{d\phi'}{\cos\phi'} = \frac{d\phi}{V^2\cos\phi} = \frac{(1-n)^2\, d\phi}{(1+n^2+2n\cos 2\phi)\cos\phi} \tag{23}$$

and separating the right-hand-side of (23) into partial fractions gives

$$\frac{d\phi'}{\cos\phi'} = \frac{d\phi}{\cos\phi} - \frac{4n\cos\phi\, d\phi}{1+n^2+2n\cos 2\phi} \tag{24}$$

Now using the standard integral result $\int \sec x\, dx = \ln\tan\left(\frac{1}{4}\pi+\frac{1}{2}x\right)$ and the useful identity linking circular and hyperbolic functions $\ln\tan\left(\frac{1}{4}\pi+\frac{1}{2}x\right) = \sinh^{-1}\tan x$ (see Appendix A) we may write

$$\sinh^{-1}\tan\phi' = \sinh^{-1}\tan\phi - \int_0^{\phi}\frac{4n\cos\phi\, d\phi}{1+n^2+2n\cos 2\phi} = \theta \tag{25}$$

We now have a function linking conformal latitude ϕ' with latitude ϕ and the integral on the right-hand-side of (25) can be evaluated using Maxima and then combined with $\sinh^{-1}\tan\phi$ to give θ. The conformal latitude ϕ' is then

$$\phi' = \tan^{-1}\left(\sinh\theta\right) \tag{26}$$

And the series for ϕ' to order n^4 is

$$\phi' = \phi + g_2\sin 2\phi + g_4\sin 4\phi + g_6\sin 6\phi + g_8\sin 8\phi + \cdots \tag{27}$$

where the coefficients $\{g_n\}$ are

$$\begin{aligned} g_2 &= -2n + \frac{2}{3}n^2 + \frac{4}{3}n^3 - \frac{82}{45}n^4 + \cdots \qquad & g_4 &= \frac{5}{3}n^2 - \frac{16}{15}n^3 - \frac{13}{9}n^4 + \cdots \\ g_6 &= -\frac{26}{15}n^3 + \frac{34}{21}n^4 + \cdots & g_8 &= \frac{1237}{630}n^4 - \cdots \end{aligned} \tag{28}$$

A direct solution for ϕ' can be obtained using isometric latitude ψ (Deakin et al. 2010) that is defined by the differential relationship that is the right-hand-side of (22)

$$d\psi = \frac{\rho\, d\phi}{\upsilon\cos\phi\, d\lambda} \tag{29}$$

with the solution (Deakin & Hunter 2010b)

$$\psi = \ln\left\{\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)\left(\frac{1-\varepsilon\sin\phi}{1+\varepsilon\sin\phi}\right)^{\varepsilon/2}\right\} \tag{30}$$

The right-hand-side of (30) may be simplified (see Appendix A) as follows

$$\begin{aligned} \psi &= \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right) - \tfrac{1}{2}\varepsilon\ln\left\{\left(1+\varepsilon\sin\phi\right)\big/\left(1-\varepsilon\sin\phi\right)\right\} \\ &= \sinh^{-1}\left(\tan\phi\right) - \varepsilon\tanh^{-1}\left(\varepsilon\sin\phi\right) \end{aligned} \tag{31}$$

and noting (22) and (25) we may write

$$\sinh^{-1}\left(\tan\phi'\right) = \sinh^{-1}\left(\tan\phi\right) - \varepsilon\tanh^{-1}\left(\varepsilon\sin\phi\right) \tag{32}$$

Taking the hyperbolic sine of both sides of (32) and simplifying gives (Karney 2011)

$$\tan\phi' = \tan\phi\sqrt{1+\sigma^2} - \sigma\sqrt{1+\tan^2\phi} \tag{33}$$

where

$$\sigma = \sinh\left\{\varepsilon\tanh^{-1}\left(\varepsilon\tan\phi\Big/\sqrt{1+\tan^2\phi}\right)\right\} \tag{34}$$

Equations (33) and (34) are given as (2) in the *Forward transformation* above.

The α-series connecting conformal latitude ϕ' and rectifying latitude μ

The α-series that is a series expression for μ as a function of ϕ' is the key to the Karney-Krueger equations (forward transformation) and a development, following Krueger (1912), is given in Deakin et al. (2010). A better alternative, suggested by Charles Karney (2010), utilizes the power of Maxima and is outlined as follows:

- Reverse the series (27) using Lagrange's theorem (see Appendix A) to give ϕ as a function of ϕ'

$$\phi = \phi' + G_2\sin 2\phi' + G_4\sin 4\phi' + G_6\sin 6\phi' + \cdots \tag{35}$$

where the coefficients $\{G_n\}$ will be functions of n

- Substitute (35) into (20) to give the α-series μ as a function of ϕ'

$$\mu = \phi' + \alpha_2 \sin 2\phi' + \alpha_4 \sin 4\phi' + \alpha_6 \sin 6\phi' + \alpha_8 \sin 8\phi' + \cdots \tag{36}$$

where the coefficients $\{\alpha_n\}$ are to order n^4

$$\begin{aligned} \alpha_2 &= \frac{1}{2}n - \frac{2}{3}n^2 + \frac{5}{16}n^3 + \frac{41}{180}n^4 - \cdots \qquad & \alpha_4 &= \frac{13}{48}n^2 - \frac{3}{5}n^3 + \frac{557}{1440}n^4 + \cdots \\ \alpha_6 &= \frac{61}{240}n^3 - \frac{103}{140}n^4 + \cdots & \alpha_8 &= \frac{49561}{161280}n^4 - \cdots \end{aligned} \tag{37}$$

Using Maxima, it is easy to extend these series to higher powers of *n*. Equation (37) are the coefficients (to order n^4) shown in the Karney-Krueger equations (4).

How is the α-series a part of the TM equations?

In the Introduction, the TM projection ($\phi, \lambda \to E, N$) is characterised as a triple projection in two parts, and by describing these parts in some detail we show how the α-series plays its vital role

First part: Gauss-Schreiber coordinates

(i) A conformal mapping of the ellipsoid (ϕ, λ) onto the conformal sphere (ϕ', λ) – see Figure 1b. This is achieved by computing ϕ' from (33), which is the 3rd step in the *Forward transformation*.

(ii) A conformal mapping of the conformal sphere to the *u*,*v* Gauss-Schreiber plane using the spherical transverse Mercator equations – Figure 1c. These equations; found in many map projection texts, e.g., Snyder (1987); have been modified slightly by Karney (2011) to give (3).

Second part: Conformal mapping from Gauss-Schreiber plane to TM plane

The scale along the central meridian of the Gauss-Schreiber projection is not constant (see Deakin et al. 2010), so a conformal mapping from the Gauss-Schreiber to the transverse Mercator plane is made with a condition that the scale along the central meridian is constant. This achieved by the following process.

A theory of conformal mapping due to Gauss allows a mapping from the *u*,*v* Gauss-Schreiber plane to the *X*,*Y* transverse Mercator plane to be represented by a complex expression <u>that we suppose to be</u>

$$Y + iX = f(u + iv) = A\left\{\xi' + i\eta' + \sum_{r=1}^{\infty} \kappa_{2r} \sin(2r\,\xi' + i\,2r\,\eta')\right\} \tag{38}$$

Y-axis is the central meridian, the *X*-axis is the equator, *A* is the rectifying radius, $\xi' = u/a$ and $\eta' = v/a$ are Gauss-Schreiber ratios and κ_{2r} are as yet unknown coefficients.

[Note that $i = \sqrt{-1}$ and the left-hand side of (38) is a complex number consisting of a real and imaginary part. The right-hand-side of (38) is a complex function of real and imaginary parameters *u* and *v* and the Cauchy-Riemann equations (Sokolnikoff & Redheffer 1966) $\partial Y/\partial u = \partial X/\partial v$ and $\partial Y/\partial v = -\partial X/\partial u$ are satisfied – a necessary condition for a conformal mapping.]

Expanding the complex trigonometric function on the right-hand-side of (38) and equating real and imaginary parts gives

$$Y = A\left\{\xi' + \sum_{r=1}^{\infty} \kappa_{2r} \sin 2r\xi' \cosh 2r\eta'\right\} \quad \text{and} \quad X = A\left\{\eta' + \sum_{r=1}^{\infty} \kappa_{2r} \cos 2r\xi' \sinh 2r\eta'\right\} \tag{39}$$

Now along the central meridian $v = 0$ (see Figure 1c), hence $\eta' = v/a = 0$ and $\cosh 2\eta' = \cosh 4\eta' = \cdots = 1$ and *Y* in (39) becomes

$$Y = A\left\{\xi' + \sum_{r=1}^{\infty} \kappa_{2r} \sin 2r\xi'\right\} = A\{\xi' + \kappa_2 \sin 2\xi' + \kappa_4 \sin 4\xi' + \kappa_6 \sin 6\xi' + \cdots\} \tag{40}$$

Also, along the central meridian $\xi' = u/a$ is an angular quantity that is identical to conformal latitude ϕ' and if the scale is unity then the *Y* coordinate is the meridian distance *M* and $Y/A = M/A = \mu$, and (40) becomes

$$\mu = \phi' + \kappa_2 \sin 2\phi' + \kappa_4 \sin 4\phi' + \kappa_6 \sin 6\phi' + \cdots \tag{41}$$

Equation (41) is identical in form to (36) and we may conclude that the coefficients $\{\kappa_{2r}\}$ are equal to the coefficients $\{\alpha_{2k}\}$ in (36) and the TM projection coordinates (to order n^6) are given by

$$X = A\left\{\eta' + \sum_{k=1}^{6} \alpha_{2k} \cos 2k\xi' \sinh 2k\eta'\right\} \qquad Y = A\left\{\xi' + \sum_{k=1}^{6} \alpha_{2k} \sin 2k\xi' \cosh 2k\eta'\right\} \tag{42}$$

These are equations (5) in the 8th step in the *Forward transformation* and so, by a judicious choice of a complex function, we see how the series for μ as a function of ϕ' with coefficients α_{2k} play their vital role.

Obtaining the α-series using Maxima

Maxima is a fully functioned Computer Algebra System (CAS) and is a derivative of Macsyma which had its origins in the 1960s at Massachusetts Institute of Technology (MIT). Macsyma was the first of the 'modern' computer algebra systems and the forerunner of programs such as Maple and Mathematica. Its development grew out of research funded by the U.S. Department of Energy (DOE) and the source code (DOE Macsyma) was maintained by William Schelter from 1982 until his death in 2001. In 1998 he obtained permission to release the Maxima source code under GNU[1] General Public License (GPL).

Maxima can be used in two modes; (i) typing simple input commands into the console screen that are acted on with the result as output printed to the console; or (ii) as a 'batch' file of instructions that are executed sequentially with output printed to the console. Batch files are the more useful way to use Maxima and the results shown in this paper have been generated from a Maxima text file 'TMseries_a.mac'. This file is available from the author and part of the Maxima output is shown in Appendix B.

To demonstrate some Maxima commands the following parts of the text file TMseries_a.mac are useful.

This part evaluates the integrand of (14) and derives the coefficients of equation (15)

```
/*******************************************************************************
   Derive the series for meridian distance M as a function of the latitude B
*******************************************************************************/
/* Integrand of the function for meridian distance */
Fintegrand:((1-n^2)^2*(1+n^2+2*n*cos(2*B))^(-3/2))$
/* expand the integrand into a Taylor series */
F:taylor(Fintegrand,n,0,maxpow)$
/* integrate the Taylor series w.r.t. latitude B */
f:integrate(F,B)$
/* reduce products and powers of sines and cosines to those of multiples */
f:trigreduce(f)$
/* expand the function */
f:expand(f)$
print("equation for MERIDIAN DISTANCE M as a function of LATITUDE B");
print("M = a/(1+n){c0*B + c2*sin(2B) + c4*sin(4B) + c6*sin(6B) + ...}");
print(" ");
/* gather coeffs of B and assign to c0 */
c0:coeff(f,B)$
/* print coeffs c0, c2, c4, c6, ... */
print("c0 = ",c0);
for i thru maxpow do
  print(c[2*i] = coeff(f,sin(2*i*B)));
```

This part evaluates the integrand of (25) and derives the coefficients of equation (27)

```
/***************************************************************************
  Derive the series for conformal latitude b as a function of the latitude B
***************************************************************************/
/* set the integrand of equation (25) */
Fintegrand:4*n*cos(B)/(1+n^2+2*n*cos(2*B))$
/* Expand the integrand and do the integration. */
F:integrate(ratdisrep(taylor(Fintegrand,n,0,maxpow)),B)$
/* Start the integral from zero.  Could have told Maxima to do a definite integral; */
/* but this leads to questions about the signs of various terms. */
F:F-subst([B=0],F)$
/* Subtract the evaluated integral from asinh(tan(B)) as per eq (25) */
```

[1] GNU is a recursive acronym for 'GNU's not Unix' chosen because GNU's design is Unix-like, but differs from Unix by being free software and containing no Unix code. GNU is a computer operating system developed by the GNU project aiming to be a complete Unix-compatible software system composed wholly of free software.

```
F:asinh(tan(B))-F$
/* Convert back to a Taylor series */
F:taylor(F,n,0,maxpow)$
/* Compute b inverting asinh(tan(B)). This magically stays as a Taylor series. */
b:atan(sinh(F))$
/* Normalize some of the trig expressions so that trigreduce() works seamlessly */
b:subst([sqrt(tan(B)^2+1)=1/cos(B),tan(B)=sin(B)/cos(B)],ratdisrep(b))$
/* Convert to multiple angle form */
b:expand(trigreduce(ratsimp(b)))$
/* Group coeffs of sin(2*i*B)
b:coeff(b,B)*B+sum(coeff(b,sin(2*i*B))*sin(2*i*B),i,1,maxpow)$
print("equation for CONFORMAL LATITUDE b as a function of LATITUDE B");
print("b = B + g2*sin(2B) + g4*sin(4B) + g6*sin(6B) + ...");
print(" ");
/* print coeffs g2, g4, g6, ... */
for i thru maxpow do
  print(g[2*i] = coeff(b,sin(2*i*B)));
```

This part reverses the series for conformal latitude given by (27). It contains a definition of the function Lagrange() which must precede the function.

```
/*******************************************************************************************
  Reverse the series b = B + g2*sin(2B) + g4*sin(4B) + ... to give B = b + G2*sin(2b) + ...
*******************************************************************************************/
/* copy b into bexpr and kill any earlier definition of b */
bexpr:b$
kill(b)$
/* reverse the series b = F(B,n) to give B = f(b,n) */
Bexpr:Lagrange(bexpr,B,b,n,maxpow)$
print("equation for LATITUDE B as a function of CONFORMAL LATITUDE b");
print("B = b + G2*sin(2b) + G4*sin(4b) + G6*sin(6b) + ...");
print(" ");
/* print coefficients G2, G4, G6, ... */
for i thru maxpow do
  print(G[2*i] = coeff(Bexpr,sin(2*i*b)));
```

The function Lagrange() that performs the series reversion is

```
/* Lagrange reversion
   reverse
      var2 = expr(var1) = series in eps
   to
      var1 = revertexpr(var2) = series in eps
   Require that expr(var1) = var1 to order eps^0.  This throws in a trigreduce to convert to
   multiple angle trig functions.
*/
Lagrange(expr,var1,var2,eps,pow):=block
([b_acc:1,B_acc:0,dB],
 dB:ratdisrep(taylor(expr-var1,eps,0,pow)),
 dB:subst([var1=var2],dB),
 for n:1 thru pow do
  (b_acc:trigreduce(ratdisrep(taylor
   (-dB*b_acc/n,eps,0,pow))),
    B_acc:B_acc+expand(diff
   (b_acc,var2,n-1))),
 var2+B_acc)$
```

Conclusion

We have demonstrated the use of mathematics in the development of the Karney-Krueger equations, some at a reasonably high level; i.e., the evaluation of elliptic integrals, and the usefulness of hyperbolic functions and complex algebra. In addition, we have given examples of Maxima 'code' that might be useful in showing how a computer algebra system can be an important tool for the geospatial scientist.

Appendix A

Taylor's theorem

This theorem, due to the English mathematician Brook Taylor (1685–1731) enables a function $f(x)$ near a point $x = a$ to be expressed from the values $f(a)$ and the successive derivatives of $f(x)$ evaluated at $x = a$. Taylor's polynomial may be expressed in the following form

$$f(x) = f(a) + (x-a) f^{(1)}(a) + \frac{(x-a)^2}{2!} f^{(2)}(a) + \frac{(x-a)^3}{3!} f^{(3)}(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n \tag{43}$$

where R_n is the remainder after *n* terms and $f^{(1)}(a)$, $f^{(2)}(a), \ldots$ etc. are first, second, ... etc. derivatives of the function $f(x)$ evaluated at $x = a$. Taylor's theorem can also be expressed as power series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \tag{44}$$

Reversion of a series

If we have an expression for a variable *z* as a series of powers or functions of another variable *y* then we may, by a reversion of the series, find an expression for *y* as series of functions of *z*. Reversion of a series can be done using Lagrange's theorem, a proof of which can be found in Bromwich (1991). Suppose that

$$y = z + xF(y) \quad \text{or} \quad z = y - xF(y) \tag{45}$$

then Lagrange's theorem states that for any function *f*

$$\begin{aligned} f(y) = f(z) &+ \frac{x}{1!} F(z) f'(z) + \frac{x^2}{2!} \frac{d}{dz}\left[\{F(z)\}^2 f'(z)\right] + \frac{x^3}{3!} \frac{d^2}{dz^2}\left[\{F(z)\}^3 f'(z)\right] + \cdots \\ &+ \frac{x^n}{n!} \frac{d^{n-1}}{dz^{n-1}}\left[\{F(z)\}^n f'(z)\right] + \cdots \end{aligned} \tag{46}$$

As an example, consider the series for rectifying latitude μ

$$\mu = \phi + d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + \cdots \tag{47}$$

And we wish to find an expression for ϕ as a function of μ.

Comparing the variables in equations (47) and (45), $z = \mu$, $y = \phi$ and $x = -1$; and if we choose $f(y) = y$ then $f(z) = z$ and $f'(z) = 1$. So, expressing (47) as

$$\mu = \phi + F(\phi) \tag{48}$$

Lagrange's theorem gives

$$\phi = \mu - F(\mu) + \frac{1}{2!} \frac{d}{d\mu}\left[\{F(\mu)\}^2\right] - \frac{1}{3!} \frac{d^2}{d\mu^2}\left[\{F(\mu)\}^3\right] + \cdots + \frac{(-1)^n}{n!} \frac{d^{n-1}}{d\mu^{n-1}}\left[\{F(\mu)\}^n\right] + \cdots \tag{49}$$

where

$$F(\phi) = d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + \cdots \quad \text{and} \quad F(\mu) = d_2 \sin 2\mu + d_4 \sin 4\mu + d_6 \sin 6\mu + \cdots$$

Hyperbolic functions

The basic functions are the hyperbolic sine of *x*, denoted by $\sinh x$, and the hyperbolic cosine of *x* denoted by $\cosh x$; they are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{50}$$

Other hyperbolic functions are in terms of these

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x} \tag{51}$$

The inverse hyperbolic function of $\sinh x$ is $\sinh^{-1} x$ and is defined by $\sinh^{-1}(\sinh x) = x$. Similarly $\cosh^{-1} x$ and $\tanh^{-1} x$ are defined by $\cosh^{-1}(\cosh x) = x$ and $\tanh^{-1}(\tanh x) = x$; both requiring $x > 0$ and as a consequence of the definitions

$$\begin{aligned} \sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) \qquad -\infty < x < \infty \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geq 1 \\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1 \end{aligned} \tag{52}$$

A useful identity linking circular and hyperbolic functions used in conformal mapping is obtained by considering the following.

Using the trigonometric addition and double angle formula we have

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right) = \ln\frac{\cos\frac{1}{2}x + \sin\frac{1}{2}x}{\cos\frac{1}{2}x - \sin\frac{1}{2}x} = \ln\frac{\left(\cos\frac{1}{2}x + \sin\frac{1}{2}x\right)^2}{\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x} = \ln\frac{1+\sin x}{\cos x} \tag{53}$$

Also, replacing x with $\tan x$ in the definition of the inverse hyperbolic functions in equations (52) we have

$$\sinh^{-1}\tan x = \ln\left(\tan x + \sqrt{1+\tan^2 x}\right) = \ln\left(\tan x + \sec x\right) = \ln\frac{1+\sin x}{\cos x} \tag{54}$$

And equating $\ln\frac{1+\sin x}{\cos x}$ from equations (53) and (54) gives

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right) = \sinh^{-1}\tan x \tag{55}$$

Appendix B

Selected Maxima output from TMseries_a.mac

```
equation for MERIDIAN DISTANCE M as a function of LATITUDE B
M = a/(1+n){c0*B + c2*sin(2B) + c4*sin(4B) + c6*sin(6B) + ...}

       4    2
      n    n
c0 =  -- + -- + 1
      64   4
        3
     3 n    3 n
c  = ---- - ---
 2    16     2
         2       4
     15 n    15 n
c  = ----- - -----
 4    16      64
           3
       35 n
c  = - -----
 6      48
          4
     315 n
c  = ------
 8    512
```

```
equation for RECTIFYING LATITUDE u as a function of CONFORMAL LAT b
u = b + alp[2]*sin(2b) + alp[4]*sin(4b) + alp[6]*sin(6b) + ...
```

$$alp_2 = \frac{41 n^4}{180} + \frac{5 n^3}{16} - \frac{2 n^2}{3} + \frac{n}{2}$$

$$alp_4 = \frac{557 n^4}{1440} - \frac{3 n^3}{5} + \frac{13 n^2}{48}$$

$$alp_6 = \frac{61 n^3}{240} - \frac{103 n^4}{140}$$

$$alp_8 = \frac{49561 n^4}{161280}$$

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